

The Kharitonov theorem and Bezoutians*

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Abstract

In this correspondence we answer a question posed in [WT99] and provide an elementary proof of the Kharitonov theorem deducing it from the classical Hermite criterion. The proof is based on the concept of a Bezoutian matrix. Generally, exploiting the special structure of such matrices (e.g., Bezoutians, Toeplitz, Hankel or Vandermonde matrices, etc.) can be interesting, e.g., leading to unified approaches in different cases, as well as to further generalizations. Here the concept of the Bezoutian matrix is used to provide a unified derivation of the Kharitonov-like theorems for the continuous-time and discrete-time settings. Finally, the (block) Anderson-Jury Bezoutians are used to propose a possible technique to attack an open difficult problem related to the robust stability in the MIMO case.

1 Introduction

1.1. Hermite's criterion. Perhaps the first solution to the polynomial stability problem was given by Hermite in his famous letter to Borchardt [H1856], and we next provide a variant of his result¹.

Theorem 1 [*Hermite*] *The polynomial*

$$p(x) = p_0 + p_1x + p_2x^2 + \cdots + p_nx^n \quad (1)$$

is stable (has all its roots in the open left-half-plane) if and only if the matrix $B = [b_{kl}]$ is positive definite, where the entries of the latter matrix are obtained from the expression

$$-\frac{j}{2} \cdot \frac{p(jx)\check{p}(jy) - \check{p}(jx)p(jy)}{x-y} = \sum_{k,l=0}^{n-1} b_{kl}x^k y^l. \quad (2)$$

Here the coefficients of

$$\check{p}(x) = p_0^* + p_1^*x + p_2^*x^2 + \cdots + p_n^*x^n \quad (3)$$

are obtained from (1) by complex conjugation.

It is worth noting that the expression on the left-hand side of (2) is a polynomial in x and y ($x-y$ cancels out) and the $n \times n$ matrix B is well-defined. Substituting

$$p(jx) = g(x) + jh(x) \quad (4)$$

into (2) one immediately sees that the above matrix B is the Bezoutian matrix

$$\text{Bez}(p) = \text{Bez}(h, g) = [b_{kl}] \quad (5)$$

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¹In fact, Hermite considered the root distribution of $f(x)$ with respect to the upper-half-plane. The root distribution with respect to the left-half plane (i.e., stability) is immediate by considering the polynomial $p(jx)$.

whose entries b_{kl} by definition are obtained from

$$\sum_{k,l=0}^{n-1} b_{kl} x^k y^l = \frac{h(x)g(y) - g(x)h(y)}{x - y}. \quad (6)$$

Hence theorem 1 can be immediately reformulated as follows.

Theorem 2 *The polynomial (1) is stable if and only if the Bezoutian matrix $Bez(h,g)$ of the two “split” polynomials $g(x)$ and $h(x)$ in (4) is positive definite.*

I.2. The Kharitonov’s theorem. Some early results on the stability of *interval polynomials* were obtained by Faedo in [F53]², but it was unclear up until 1978 how to efficiently check the stability of such an infinite set of *interval polynomials* (clearly running of an infinite set of stability tests is not feasible in practice). In [K78] Kharitonov obtained the following fundamental result.

Theorem 3 [Kharitonov] *The infinite set of all polynomials of the form (1) whose coefficients lie in prescribed intervals $\underline{p}_i \leq p_i \leq \bar{p}_i$ is stable if and only if the following four “boundary” polynomials are stable:*

$$\begin{aligned} k_1(x) &= \hat{e}(x) + \hat{o}(x), & k_2(x) &= \hat{e}(x) + \check{o}(x), \\ k_3(x) &= \check{e}(x) + \hat{o}(x), & k_4(x) &= \check{e}(x) + \check{o}(x), \end{aligned} \quad (7)$$

where

$$\begin{aligned} \hat{e}(x) &= \underline{p}_0 + \bar{p}_2 x^2 + \underline{p}_4 x^4 + \bar{p}_6 x^6 + \dots, \\ \check{e}(x) &= \bar{p}_0 + \underline{p}_2 x^2 + \bar{p}_4 x^4 + \underline{p}_6 x^6 + \dots, \\ \hat{o}(x) &= \underline{p}_1 x + \bar{p}_3 x^3 + \underline{p}_5 x^5 + \bar{p}_7 x^7 + \dots, \\ \check{o}(x) &= \bar{p}_1 x + \underline{p}_3 x^3 + \bar{p}_5 x^5 + \underline{p}_7 x^7 + \dots, \end{aligned}$$

Clearly, the theorem 3 is equivalent to the following one.

Theorem 4 [Kharitonov-Hermite] *The matrix $Bez(p)$ is positive definite if and only if only four Bezoutians $Bez(k_1), Bez(k_2), Bez(k_3), Bez(k_4)$ of the four polynomials in (7) are positive definite.*

I.3. A connection between the Kharitonov and Hermite’s results. There is a vast literature on the Kharitonov’s theorem, focusing mainly on two issues. The first is constructing elementary proofs for the Kharitonov theorem (see, e.g., [B87],[V88],[WT99]) and the second is addressing its counterparts and generalizations. Typically the motivation for the first direction is to provide more insights into the connections of theorem 3 to classical results. Surprisingly enough, a connection between Kharitonov’s theorem 3 and the classical Hermite theorem 1 was never fully elaborated.

Moreover, Willems and Tempo [WT99] asked recently if it is possible to provide a direct proof for the theorem 4. One should mention that a brute-force approach does not work here since examples show that the matrix $B(p) - B(k_i)$ may not be positive definite for any of the four k_i ’s.

I.4. The structure of this correspondence. In the second section of our correspondence we answer the Willems-Tempo question and specify a proof of the theorem 4 based on the congruence of Bezoutians to particular diagonal matrices which can be analyzed further.

Structured matrices (of which Bezoutians are a special case) have attracted some attention recently (see, e.g., a collection of papers [O01] and many references therein). Typically their use may offer the following two advantages. First, structured matrix formulations may lead to unified treatments of various special cases. Secondly, the matrix language can be used to formulate natural matrix generalizations. In the second section of this note we show that with little modifications the continuous-time results of the first section can be carried over to the discrete-time case. In particular, the results of [V88] follow from the properties of discrete-time Bezoutians. In this way, the Bezoutian approach reveals a beautiful analogy between the continuous and discrete cases.

All the above results were limited to the SISO case. In the fourth section we use the concept of the more general Anderson-Jury Bezoutian, and ask if it is possible to use its properties to obtain a solution to the Kharitonov-like problems in the MIMO case.

²The authors are grateful to Prof. Dario A.Bini of the University of Pisa for making a copy of [F53] available to us.

2 Continuous-time case

II.1. Properties of Bezoutians. To deduce theorem 4 from theorem 1 we need to establish the following elementary but crucial property of positive definite Bezoutian matrices.

Lemma 5 *If the Bezoutian $B = \text{Bez}(h, g)$ defined by (6) of two real polynomials is positive definite then the following five conditions hold.*

1. The roots of $g(x)$ and $h(x)$ are all real.
2. The roots of each of the $h(x)$ and $g(x)$ are distinct.
3. Let us denote

$$\begin{cases} a(x) \triangleq h(x), & b(x) \triangleq g(x) & \text{if } \deg h(x) \geq \deg g(x); \\ a(x) \triangleq g(x), & b(x) \triangleq -h(x) & \text{if } \deg h(x) < \deg g(x); \end{cases} \quad \text{Then}$$

$$VBV^* = \begin{bmatrix} a'(x_1)b(x_1) & & & \\ & 0 & \ddots & \\ & & & 0 \\ & & & & a'(x_n)b(x_n) \end{bmatrix} \quad (8)$$

where V is the Vandermonde matrix $V_{ik} = [x_i^{k-1}]$ whose nodes $\{x_i\}$ are the roots of $a(x)$.

4. The roots of $g(x)$ and $h(x)$ interlace.
5. The leading coefficients of $a(x)$ and $b(x)$ have the same sign.

The conditions 1), 2), 4) and 5) are also sufficient for the positive definiteness of B .

The proof is provided in the appendix below. We are now ready to prove theorem 4.

II.2. A Bezoutian Proof of the Hermite-Kharitonov theorem.

proof Let us split polynomials $k_i(jx) = g_i(x) + jh_i(x)$ defined in (7) similarly to the splitting of $p(jx)$ in (4). Since all four matrices $B(h_i, g_k)$ are positive definite, all four pairs of polynomials $\{g_i(x), h_k(x)\}$ have the five properties stated in lemma 5 above. We now have to prove that the pair $\{g, h\}$ has the same properties so that lemma 5 could be applied to show that $B(h, g)$ is positive definite.

First note that (7) imply that for all $x \in R$ we have

- (a) Either $g_1(x) \leq g(x) \leq g_4(x)$ or $g_1(x) \geq g(x) \geq g_4(x)$.
- (b) Either $h_1(x) \leq h(x) \leq h_4(x)$ or $h_1(x) \geq h(x) \geq h_4(x)$.

Hence, the roots of $g(x)$ and $h(x)$ are real (property 1)), distinct (property 2)). One can also see that the leading coefficients of $g(x)$ and $h(x)$ inherit their signs from $g_i(x)$ and $h_i(x)$ so that the property 5) also holds.

It remains to show the property 4). To this end we notice that the properties (a) - (b) above imply

$$\text{Either } z_{i,g_1} \leq z_{i,g} \leq z_{i,g_4}, \text{ or } z_{i,g_1} \geq z_{i,g} \geq z_{i,g_4}. \quad (9)$$

$$\text{Either } z_{i,h_1} \leq z_{i,h} \leq z_{i,h_4}, \text{ or } z_{i,h_1} \geq z_{i,h} \geq z_{i,h_4}. \quad (10)$$

where z_{i,g_1} denotes the i 'th root of $g_1(x)$ and the other quantities involved are defined similarly.

Now, the roots of each $g_1(x)$ and $g_4(x)$ interlace with the roots of each $h_1(x)$ and $h_4(x)$. Hence it follows (9) that the roots of g interlace with the roots of each $h_1(x)$ and $h_4(x)$. Secondly, employing (10) one sees that the roots of $g(x)$ and $h(x)$ interlace (property 4)). QED.

This answers the question posed in [WT99] and provides a direct matrix proof for the theorem 4.

One of the properties in lemma 5 is the interlacing property 4). One has to mention that it was used earlier in some direct proofs of the Kharitonov's theorem 3, see, e.g., [B87]. One advantage of its use in the context of the factorization of Bezoutians (8) is that it provides a beautiful unified approach to the continuous time and discrete time cases, the latter to be discussed in the next section. Moreover, since the concept of the Bezoutian has a natural counterpart for matrix polynomials, it provides one possible way to approach the problem of generalizing Kharitonov's theorem to the MIMO case.

3 Discrete-time case

III.1. Discrete-time splitting, symmetric polynomials and Bezoutians. We start with recalling the discrete-time analogues of the results described in the Introduction. The discrete-time counterpart of the Hermite's theorem 1 is the following.

Theorem 6 [Schur] *The polynomial (1) is discrete-time stable (has all its roots in the interior of the unit disk \mathbb{T}) if and only if the matrix $B = [b_{kl}]$ is positive definite, where the entries of the latter matrix are obtained from the expression*

$$\frac{1}{2} \cdot \frac{p^\#(x)\check{p}^\#(y) - p(x)\check{p}(y)}{1 - xy} = \sum_{k,l=0}^{n-1} b_{kl}x^k y^l. \quad (11)$$

with $\check{p}(x)$ is defined as in (3) and $p^\#(x)$ is defined by

$$p^\#(x) = x^n \check{p}\left(\frac{1}{x}\right) = p_0^* x^n + p_1^* x^{n-1} + \dots + p_n^*. \quad (12)$$

This theorem was proven in [S18], used in [C22] and generalized in [F26]. Let us now define splitting of $p(x)$. Following [K33] a polynomial $p(x)$ is called symmetric if $p^\#(x) = p(x)$. An analogue of (4) in the discrete-time case is played by two symmetric polynomials $\{g(x), h(x)\}$ obtained from

$$p(x) = g(x) + jh(x), \quad p^\#(x) = g(x) - jh(x). \quad (13)$$

Clearly, $g(x) = \frac{p(x)+p^\#(x)}{2}$ and $h(x) = \frac{p(x)-p^\#(x)}{2j}$.

Substituting (13) into (11) one immediately sees that the above matrix B is the (discrete-time) Bezoutian matrix

$$\text{Bez}_{\mathbb{T}}(p) = \text{Bez}_{\mathbb{T}}(h, g) = [b_{kl}] \quad (14)$$

whose entries b_{kl} by definition are obtained from

$$\sum_{k,l=0}^{n-1} b_{kl}x^k y^l = j \frac{g(x)\check{h}(y) - h(x)\check{g}(y)}{1 - xy}, \quad (15)$$

where $\check{p}(x)$ is defined as in (3). Hence theorem 6 can be immediately reformulated as follows.

Theorem 7 *The polynomial (1) is stable in the discrete time sense if and only if the Bezoutian matrix $\text{Bez}_{\mathbb{T}}(h, g)$ of the two "split" polynomials $g(x)$ and $h(x)$ in (13) is positive definite.*

Before studying in the next section discrete-time Bezoutians we need to discuss some elementary properties of symmetric polynomials $a(x) = a^\#(x) = \sum_{k=0}^n a_k x^k$.

- One immediate property is: $a_k = a_{n-k}^*$.
- The roots of symmetric polynomials are symmetric with respect to the unit circle. Indeed, (12) implies $a(z) = 0 \Rightarrow a\left(\frac{1}{z^*}\right) = 0$.
- Let $a_0 \neq 0$ and $a_n \neq 0$, and n is even. Then the corresponding Laurent polynomial

$$\frac{a(x)}{x^{\frac{n}{2}}} = \sum_{k=-\frac{n}{2}}^{\frac{n}{2}} \tilde{a}_k z^k \quad \text{with} \quad \tilde{a}_k = \begin{cases} a_{k+\frac{n}{2}} & \text{if } k \geq 0; \\ (\tilde{a}_{-k})^* & \text{if } k < 0. \end{cases}$$

clearly assumes real values on the unit circle. Hence one can associate with $a(x) = a^\#(x)$ a real function

$$\tilde{a}(\theta) = \frac{a(e^{j\theta})}{e^{jn\theta/2}} = \sum_{k=-\frac{n}{2}}^{\frac{n}{2}} \tilde{a}_k e^{jk\theta} \quad (16)$$

- If n is odd then we define the corresponding

$$\tilde{a}(\theta) = \frac{a(e^{j\theta})}{e^{jn\theta/2}} = \sum_{k=1}^{\frac{n+1}{2}} (\tilde{a}_{-k} e^{-j\frac{(2k-1)}{2}\theta} + \tilde{a}_k e^{j\frac{(2k-1)}{2}\theta}) \quad (17)$$

$$\text{where } \tilde{a}_k = \begin{cases} a_{k+\frac{n-1}{2}} & \text{if } k > 0; \\ (\tilde{a}_{-k})^* & \text{if } k < 0. \end{cases}$$

III.2. Properties of discrete-time Bezoutians. The following lemma is a discrete-time counterpart of lemma 5.

Lemma 8 *If the Bezoutian $\text{Bez}_{\mathbb{T}}(h, g)$ (6) of two symmetric polynomials of the same degree is positive definite then the following five conditions hold.*

1. The roots of $g(x)$ and $h(x)$ all lie on the unit circle.
2. The roots of each of the $h(x)$ and $g(x)$ are distinct, and the degree of $h(x)$ and $g(x)$ is the same.
3. Let $\{x_k = e^{j\theta_k}\}$ be the roots of $h(x)$, and $V = [x_i^{j-1}]$. Then

$$VBV^* = j \begin{bmatrix} \frac{h'(x_1)g(x_1)}{x_1^{n-1}} & & & \\ 0 & \ddots & & 0 \\ & & \frac{h'(x_n)g(x_n)}{x_1^{n-1}} & \\ & & & \end{bmatrix} = \quad (18)$$

$$\begin{bmatrix} \tilde{h}'(\theta_1)\tilde{g}(\theta_1) & & & \\ 0 & \ddots & & 0 \\ & & \tilde{h}'(\theta_n)\tilde{g}(\theta_n) & \\ & & & \end{bmatrix}. \quad (19)$$

4. The roots of $g(x)$ and $h(x)$ interlace on the unit circle. The same is true for $\tilde{g}(\theta)$ and $\tilde{h}(\theta)$ on the interval $\theta \in [0, 2\pi]$.
5. The quantities $\tilde{h}'(0)$ and $\tilde{g}(0)$ should have the same sign.
6. .

The conditions 1), 2), 4) and 5) are also sufficient for the positive definiteness of B .

The proof is provided in the appendix below.

III.3. A Bezoutian proof of Vaidyanathan's Theorem. It is known that there is no direct counterpart of Kharitonov's theorem for the discrete case. One reason for this can be seen by comparing Lemma 5 with Lemma 8. In the continuous time case the positive definiteness of the Bezoutian reduces to statements about the polynomials $g(x)$ and $h(x)$ living in the original "coefficient space," as Lemma 5 shows. On the other hand, in the discrete time case the positive definiteness reduces to statements about the, so to speak, "processed" polynomials $\tilde{g}(\theta)$ and $\tilde{h}(\theta)$, see, e.g., (19). Therefore, the discrete time analogue of Kharitonov's theorem is more naturally formulated in terms of the coefficients of $\tilde{g}(\theta)$ and $\tilde{h}(\theta)$. To formulate the corresponding result by Vaidyanathan we need to express $\tilde{g}(\theta)$ and $\tilde{h}(\theta)$ in a more appropriate form needed later to define the discrete-time "interval polynomials."

Lemma 9 *Let $p(x) = g(x) + jh(x)$, with symmetric $g(x)$ and $h(x)$. The functions $\tilde{g}(\theta)$ and $\tilde{h}(\theta)$ can be represented as follows:*

$$\tilde{g}(\theta) = \sum_k c_{2k}^{(p)} \cos^{2k}\left(\frac{\theta}{2}\right) \quad (20)$$

$$\tilde{h}(\theta) = \sin(\theta) \sum_k d_{2k}^{(p)} \cos^{2k}\left(\frac{\theta}{2}\right) \quad (21)$$

The proof of this is in the appendix(c.f. with [V88]). The next analogue of Kharitonov's theorem [V88] uses these new coefficients c_k and d_k .

Theorem 10 [Vaidyanathan]

Let $\{g_1(x), g_2(x), h_1(x), h_2(x)\}$ be four symmetric polynomials such that four related polynomials polynomials $k_{mp}(x) = g_m(x) + jh_p(x)$, $(m, p = 1, 2)$ have real coefficients, are stable in the discrete-time sense, and satisfy

$$c_m^{(k_{11})} \leq c_i^{(k_{22})}, \quad d_i^{(k_{11})} \leq d_i^{(k_{22})}$$

(where c_i and d_i are defined similarly to (20) and (21)).

Then all "interval" polynomials $p(x)$ satisfying

$$c_i^{(k_{11})} \leq c_i^{(p)} \leq c_i^{(k_{22})} \tag{22}$$

$$d_i^{(k_{11})} \leq d_i^{(p)} \leq d_i^{(k_{22})} \tag{23}$$

are stable in the discrete-time sense.

We shall prove the following immediate reformulation of the latter theorem.

Theorem 11 [Schur-Vaidyanathan] The positive definiteness of the four Bezoutians $Bez(h_k, g_m)$ where $k, m = 1, 2$ implies the positive definiteness of $Bez(h, g)$ where h and g are constrained by equations (22) and (23).

Proof.

Exploiting lemma 9 we can naturally adapt the arguments made in the continuous time case.

The fact that the four Bezoutians $Bez(h_1, g_1), Bez(h_1, g_2), Bez(h_2, g_1), Bez(h_2, g_2)$ are all positive definite implies stability of the four polynomials k_{mp} .

We first note that (22) and (23) imply that for all $\theta \in [0, 2\pi]$ we have

(a) Either $\tilde{g}_1(\theta) \leq \tilde{g}(\theta) \leq \tilde{g}_2(\theta)$ or $\tilde{g}_1(\theta) \geq \tilde{g}(\theta) \geq \tilde{g}_2(\theta)$.

(b) Either $\tilde{h}_1(\theta) \leq \tilde{h}(\theta) \leq \tilde{h}_2(\theta)$ or $\tilde{h}_1(\theta) \geq \tilde{h}(\theta) \geq \tilde{h}_2(\theta)$.

Hence, the roots of $g(x)$ and $h(x)$ all lie on the unit circle (property 1)), and are distinct (property 2)).

We need to show property 4). To this end we notice that the properties (a) - (b) above imply

$$\text{Either } \theta_{i,g_1} \leq \theta_{m,g} \leq \theta_{i,g_2}, \text{ or } \theta_{i,g_1} \geq \theta_{i,g} \geq \theta_{i,g_2}. \tag{24}$$

$$\text{Either } \theta_{i,h_1} \leq \theta_{i,h} \leq \theta_{i,h_2}, \text{ or } \theta_{i,h_1} \geq \theta_{i,h} \geq \theta_{i,h_2}. \tag{25}$$

where θ_{i,g_1} denotes the i 'th root of $\tilde{g}_1(\theta)$ and the other quantities involved are defined similarly.

Now, the roots of each $\tilde{g}_1(\theta)$ and $\tilde{g}_2(\theta)$ interlace with the roots of each $\tilde{h}_1(\theta)$ and $\tilde{h}_2(\theta)$. Hence it follows (24) that the roots of $\tilde{g}(\theta)$ interlace with the roots of each $\tilde{h}_1(\theta)$ and $\tilde{h}_2(\theta)$. Secondly, employing (25) one sees that the roots of $\tilde{g}(\theta)$ and $\tilde{h}(\theta)$ interlace (property (4)).

As for property (5), it follows that the bounds on the coefficients (22) and (23) imply that the signs at 0 of $\tilde{h}'(\theta)$ and $\tilde{g}(\theta)$ are inherited from the signs at 0 of $\tilde{h}'_i(\theta)$ and $\tilde{g}_i(\theta)$, $i = 1, 2$. Property 5 follows from this. QED.

4 The MIMO Case: the Anderson-Jury Bezoutian and Lerer-Tismenetsky results

We conclude this note with an open question about robust stability in the MIMO case. Let

$$L(\lambda) = A_0 + A_1\lambda + \dots + A_n\lambda^n$$

be a matrix polynomial with A_k be $m \times m$ matrices. We refer, e.g., to [GLR82] to the theory of matrix polynomials, and here only briefly recall some definitions. A number μ is called an eigenvalue of $L(\lambda)$ if $\det L(\mu) = 0$. Clearly, if A_n is nonsingular $L(\lambda)$ has nm eigenvalues. A matrix polynomial is called

stable if all of its eigenvalues lie in the open left half plane. A matrix polynomial is called stable in the discrete-time case if all its eigenvalues lie in the interior of the unit disk. The natural counterparts of theorems 3 and 10 can be immediately formulated. For example, one can ask if stability of the entire set of matrix polynomials satisfying

$$\underline{A}_k \leq A_k \leq \overline{A}_k$$

could be reduced to checking a finite number of matrix polynomials.

In the rest of the paper we describe one approach that might be useful to attack this open problem. It is based on the concept of the Anderson-Jury Bezoutian, and some results of Lerer and Tismenetsky.

We proceed with definitions. Let $P_1(\lambda) = \sum_{i=0}^n \lambda^i M_i$ and $P_2(\lambda) = \sum_{i=0}^n \lambda^i M'_i$, where M_i and M'_i are matrices. Certainly, $P_1(\lambda)$ and $P_2(\lambda)$ have a common multiple, that is, there exist matrix polynomials $Q_1(\lambda)$ and $Q_2(\lambda)$ such that $Q_1(\lambda)P_1(\lambda) = Q_2(\lambda)P_2(\lambda)$. Since $P_1(\lambda)$ and $P_2(\lambda)$ have the same degree, it follows that $Q_1(\lambda)$ has the same degree as $Q_2(\lambda)$. Denote the degree of Q_1 and Q_2 as m .

The Anderson-Jury Bezoutian associated with (P_1, Q_1, P_2, Q_2) is defined as follows:

$$\tilde{B}(\lambda, \mu) = \frac{Q_1(\lambda)P_1(\mu) - Q_2(\lambda)P_2(\mu)}{\lambda - \mu} = \sum_{i,k=0}^{m-1, n-1} \Gamma_{ik} \lambda^i \mu^k$$

The usefulness of the Anderson-Jury Bezoutian in studying stability of matrix polynomials was established in in [LT82] where the following result was proven.

Theorem 12 [Lerer-Tismenetsky] *Let $L(\lambda)$ be a matrix polynomial. Let us assume say that the following left-right factorization holds*

$$L^*(-\lambda)L(\lambda) = L_1^*(-\lambda)L_1(\lambda). \quad (26)$$

Let $L_0(\lambda)$ be the greatest common divisor of $L(\lambda)$ and $L_1(\lambda)$. Let \tilde{B} be the Bezoutian generated by $(L(\lambda), L^*(-\lambda), L_1(\lambda), L_1^*(-\lambda))$ and let $\hat{B} = \tilde{B} \cdot \text{diag}(I, -I, \dots, (-1)^{n-1}I)$. Let $\gamma_+(L)$ be the number of eigenvalues of L in the left half plane, and let $\pi(\hat{B})$ be the number of positive squares of the the matrix \hat{B} . Then,

$$\gamma_+(L) = \pi(\hat{B}) + \gamma_+(L_0)$$

There are many factorizations of the form (26). If $L(\lambda)$ is stable, then $L_1(\lambda)$ can be chosen to be anti-stable (all eigenvalues are in the right-half plane) and in this case their greatest common divisor $L_0 = I$. This implies that

$$\gamma_+(L) = \pi(\hat{B})$$

i.e., a stable polynomial corresponds to a positive definite Bezoutian form. There is a natural counterpart of theorem 12 for the discrete-time case, see [LT82], theorem 2.10, where the positive definiteness of the Bezoutian is connected with the discrete time stability of the matrix polynomial.

We therefore ask if the properties of the above generalized Bezoutians can be exploited to obtain Kharitonov-type theorems for matrix polynomials, both for the continuous and discrete cases.

5 Appendix: Proofs of Three Lemmas

5.1 Proof of Lemma 5

1. The condition $B > 0$ implies $uBu^* > 0$ for any row vector u . Assuming z is not real, let us evaluate

$$\begin{bmatrix} 1 & z & \dots & z^{n-2} & z^{n-1} \end{bmatrix} B \begin{bmatrix} 1 \\ z^* \\ \vdots \\ (z^*)^{n-2} \\ (z^*)^{n-1} \end{bmatrix} = \quad (27)$$

$$B(z, z^*) = \frac{a(z)b(z^*) - a(z^*)b(z)}{z - z^*}. \quad (28)$$

If z is a non-real zero of one of the real polynomials $a(x)$ or $b(x)$ then the expression in (28) must be zero, which would mean that B is not positive definite. Hence all $\{x_i\}$ must be real.

2. Let us now evaluate the expression in (27) for z being a real root of $a(x)$. Clearly, it is equal to

$$B(z, z^*) = \lim_{y \rightarrow z} \frac{a(z)b(y) - a(y)b(z)}{z - y} = a'(z)b(z) \quad (29)$$

Thus, if z would be a multiple root of $a(x)$ the expression in (29) would be zero. Then an argument similar to the one above would imply that the matrix B is not positive definite. Hence all roots of $a(x)$ are simple.

3. Since V stacks the rows of the form $[1 \quad x_i \quad \cdots \quad x_i^n]$ shown in (27) we have $VBV^* = [B(x_i, x_k^*)]$. Evaluating the entries of the latter matrix and using (29) one obtains (8).
4. Since all the roots $\{x_i\}$ of $a(x)$ are simple, hence the sequence $\{a'(x_i)\}$ is sign-interchanging. Positive definiteness of B and (8) imply that $\{b(x_i)\}$ must be sign-interchanging as well. Hence the zeros of $a(x)$ and $b(x)$ interlace. In particular, this means that the roots of $b(x)$ are simple as well.
5. Finally, all the elements $a'(x_i)b(x_i)$ of the matrix on the right-hand side of (8) must be positive. Hence the leading coefficients of $a(x)$ and $b(x)$ should have the same sign.

After the above discussion the sufficiency of 1), 2), 4) and 5) is obvious.

5.2 Proof of Lemma 8

1. The condition $B > 0$ implies $uBu^* > 0$ for any row vector u . Assuming $z \neq \frac{1}{z^*}$, i.e., it is not on the unit circle, let us evaluate

$$\left[1 \quad z \quad \cdots \quad z^{n-2} \quad z^{n-1} \right] B \begin{bmatrix} 1 \\ z^* \\ \vdots \\ (z^*)^{n-2} \\ (z^*)^{n-1} \end{bmatrix} = \quad (30)$$

$$B(z, z^*) = j \frac{g(z)\check{h}(z^*) - h(z)\check{g}(z^*)}{1 - zz^*}. \quad (31)$$

If $z \neq \frac{1}{z^*}$ is a zero of one of the polynomials $g(x)$ or $h(x)$ then the expression in (31) must be zero, but this would mean that B is not positive definite. Hence all $\{x_i\}$ must lie on the unit circle.

2. Let us now evaluate the expression in (30) for the z being a unimodular root of $h(x)$. Clearly, it is equal to

$$B(z, z^*) = \lim_{y \rightarrow z^*} j \frac{g(z)\check{h}(y) - h(z)\check{g}(y)}{1 - zy}. \quad (32)$$

Making the substitution $y = \frac{1}{u}$ and using the fact that both $h(x)$ and $g(x)$ are symmetric, one can show that (32) reduces to

$$(z^*)^{n-1} \lim_{u \rightarrow z} j \frac{g(z)h(u) - h(z)g(u)}{u - z} = j \frac{h'(z)g(z)}{(z)^{n-1}} \quad (33)$$

Thus, if z would be a multiple root of $h(x)$ the expression in (33) would be zero. Then an argument similar to the one above would imply that the matrix B is not positive definite. Hence all roots of $h(x)$ are simple.

According to Schur's theorem, the positive definiteness of $\text{Bez}(h, g)$ implies that $p(x) = h(x) + jg(x)$ is stable. Letting $p(x) = p_0 + \cdots + p_n x^n$ it can be seen that the stability of $p(x)$ implies that $|p_0| \neq |p_n|$. From this it follows that $\deg g(x) = \deg h(x) = \deg p(x)$.

3. Since V stacks the rows of the form $[1 \ x_i \ \cdots \ x_i^n]$ shown in (30) we have $VBV^* = [B(x_i, x_k^*)]$. Evaluating the entries of the latter matrix and using (33) one obtains (18).

To obtain (19) let us look closely at the expression on the right-hand-side of (33). Using $\frac{\partial h}{\partial x} = \frac{\partial h}{\partial \theta} \frac{\partial \theta}{\partial x} = \frac{\partial h}{\partial \theta} \frac{-i}{x}$ we see that it is equal to $\frac{h'(\theta)}{e^{j\theta n/2}} \cdot \frac{g(\theta)}{e^{j\theta n/2}}$. Using the definitions (16) and (17) and the fact that x is a root of $h(x)$ (so that θ is the root of \tilde{h} we obtain (19).

4. Since all the roots $\{x_i\}$ of $h(x)$ are simple, hence the sequence $\{\tilde{h}'(\theta_i)\}$ is sign-interchanging. Positive definiteness of B and (19) imply that $\{\tilde{g}(\theta_i)\}$ must be sign-interchanging as well. Hence the zeros of $\tilde{h}(\theta)$ and $\tilde{g}(\theta)$ interlace. In particular, this means that the roots of $\tilde{g}(\theta)$ are simple as well.
5. Finally, all the elements $\tilde{h}'(\theta_i)\tilde{g}(\theta_i)$ of the matrix on the right-hand side of (19) must be positive. Because 0 is always a root of $\tilde{h}(\theta)$ according to (21), it follows that $\tilde{h}'(0)$ and $\tilde{g}(0)$ must have the same sign.

After the above discussion the sufficiency of 1), 2), 4) and 5) is obvious.

5.3 Proof of Lemma 9

1. Following [V88] construct the representation for $\tilde{g}(\theta)$ shown in (16), i.e $\tilde{g}(\theta) = \sum_{k=-\frac{n}{2}}^{\frac{n}{2}} \tilde{g}_k e^{jk\theta}$ From the construction of $g(x)$ we know that $\tilde{g}_{-k} = \tilde{g}_k$, and therefore one can express $\tilde{g}(\theta)$ as a sum of cosines: $\tilde{g}(\theta) = \sum_{k=0}^{\frac{n}{2}} 2\tilde{g}_k \cos(k\theta)$. Furthermore, $\cos(k\theta)$ is a linear combination $\cos \theta, \cos^2 \theta, \dots, \cos^k \theta$. Using this, one can express $\tilde{g}(\theta)$ as a linear combination of the powers of cosine: $\tilde{g}(\theta) = \sum_{k=0}^{\frac{n}{2}} \tilde{c}_k \cos^k \theta$, where \tilde{c}_k are the transformed coefficients. Now one can use the identity $\cos(\theta) = 2 \cos^2(\frac{\theta}{2}) - 1$ to represent $\tilde{g}(\theta)$ as an expansion of the even powers of $\cos(\frac{\theta}{2})$, i.e. $\tilde{g}(\theta) = \sum_{k=0}^{\frac{n}{2}} c_{2k} \cos^{2k}(\frac{\theta}{2})$.
2. The same can be done for $\tilde{h}(\theta)$. Since $\tilde{h}_{-k} = -\tilde{h}_k$, we can similarly represent $\tilde{h}(\theta)$ as a combination of sines: $\tilde{h}(\theta) = \sum_{k=1}^{\frac{n}{2}} \tilde{d}_k \sin k\theta$. Note that there is no term associated with $k = 0$. Because of this, it is obvious that one can represent $\tilde{h}(\theta)$ as follows: $\tilde{h}(\theta) = \sin(\theta) \cdot \sum_k \hat{d}_k \cos^k \theta$. Then, one proceeds the same way as before to change the the sum to a sum of the even powers of cosine: $\tilde{h}(\theta) = \sin(\theta) \sum_k d_{2k} \cos^{2k}(\frac{\theta}{2})$.

In both cases, this representation clearly always exists; moreover, since each step was reversible, it is unique.

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