A GENERALIZED KHARITONOV THEOREM FOR QUASI-POLYNOMIALS, ENTIRE FUNCTIONS, AND MATRIX POLYNOMIALS*

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Abstract. The classical Kharitonov theorem on interval stability cannot be carried over from polynomials to arbitrary entire functions. In this paper we identify a class of entire functions for which the desired generalization of the Kharitonov theorem can be proven. The class is wide enough to include quasi-polynomials occurring in the study of retarded systems with time delays. We also derive results for matrix polynomials and matrix entire functions.

1. Introduction.

1.1. Polynomial stability and the classical Kharitonov theorem.

Polynomial stability problems of various types arise in a number of problems in mathematics and engineering. Perhaps the first solution to the polynomial stability problem was given by Hermite in his famous letter to Borchardt [H1856]. Hermite showed that a polynomial

$$F(x) = f_0 + f_1 x + f_2 x^2 + \cdots + f_n x^n \tag{1.1}$$

is stable, i.e., all its roots lie in the open left-half-plane, if and only if what we now call a Bezoutian matrix [associated with $F(x)$] is positive definite[1].

A decade later the problem has attracted a close attention of mechanical engineers who faced the problem of resolving the instability of steam engines. Motivated by his studies[2] in [M1868] J.C.Maxwel, being totally unaware of the Hermite result, posed at a meeting of the London Mathematical Society in 1868 an open problem of finding a method for checking if a polynomial in (1.1) is stable, which motivated the Adams prize competition at Cambridge (1875). The prize was won by Routh who solved the problem in [R1977], and his algorithm was given a different shape by Hurwitz [H1895].

However, in practice nothing can be measured exactly, so often one has to deal with some *"uncertain"* families of polynomials, e.g., *interval polynomials* of the form (1.1) with the coefficients living in certain prescribed intervals

$$\underline{f_i} \le f_i \le \bar{f}_i. \tag{1.2}$$

Some early results on the stability of such interval polynomials were obtained by Faedo in [F53][3], but it was unclear up until 1978 how to efficiently check the stability of such an infinite set of *interval polynomials* (clearly running of an infinite set of stability tests is not feasible in practice). In [K78] Kharitinov obtained the following fundamental result showing that the problem can be solved by just running the Routh-Hirwitz test for only four "border" polynomials.

THEOREM 1.1. *[Kharitonov] All polynomials (1.1) satisfying (1.2) are stable if and only*

*This work was supported in part by the NSF contracts 0242518 and 0098222.

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[1]Actually, Hermite discussed the localization of the roots in the upper half plane, but a reformulation of his result to $F(-jz)$ and so for the left half plane case (i.e., stability) is trivial.

[2]J.C.Maxwell found stability criteria for a polynomial of degree 3.

[3]The authors are grateful to Prof. Dario A.Bini of the University of Pisa for making a copy of [F53] available to us.

if the following four polynomials are stable:

$$\begin{aligned} F_{min,max}(x) &= F_{e,min}(x) + F_{o,max}(x), \\ F_{min,min}(x) &= F_{e,min}(x) + F_{o,min}(x), \\ F_{max,max}(x) &= F_{e,max}(x) + F_{o,max}(x), \\ F_{max,min}(x) &= F_{e,max}(x) + F_{o,min}(x), \end{aligned} \tag{1.3}$$

where

$$\begin{aligned} F_{e,min}(x) &= \underline{f}_0 + \overline{f}_2 x^2 + \underline{f}_4 x^4 + \overline{f}_6 x^6 + \dots, \\ F_{e,max}(x) &= \overline{f}_0 + \underline{f}_2 x^2 + \overline{f}_4 x^4 + \underline{f}_6 x^6 + \dots, \\ F_{o,max}(x) &= \underline{f}_1 x + \overline{f}_3 x^3 + \underline{f}_5 x^5 + \overline{f}_7 x^7 + \dots, \\ F_{o,min}(x) &= \overline{f}_1 x + \underline{f}_3 x^3 + \overline{p}_5 x^5 + \underline{f}_7 x^7 + \dots, \end{aligned}$$

REMARK 1. *The notations in the Kharitonov's theorem have the following meaning. If we partition*

$$F(w) = \underbrace{F_e(w)}_{\textit{even terms}} + \underbrace{F_o(w)}_{\textit{odd terms}},$$

then

$$F_{e,min}(jw) \le F_e(jw) \le F_{e,max}(jw), \qquad (w \in \mathbb{R}), \tag{1.4}$$

$$\frac{F_{o,min}(jw)}{jw} \le \frac{F_o(jw)}{jw} \le \frac{F_{o,max}(jw)}{jw}, \qquad (w \in \mathbb{R}). \tag{1.5}$$

The latter theorem was immediately followed by a vast literature; the result has been generalized in many ways, and it enjoyed a number of applications in mechanical and electrical engineering.

1.2. Entire functions, quasi-polynomials, and stability. Stability problems for polynomials were intensively studied for many decades, and by now they are quite well understood. Similar problems for entire functions appear in several applications. However, they are typically much more involved and much more challenging. Let us first consider a very transparent scalar example.

EXAMPLE 1. *Consider a differential equation with a time delay τ, i.e.,*

$$\frac{dy}{dt} = zy, \qquad y(t) + \beta y(t - \tau) = 0. \tag{1.6}$$

The solution clearly has the form

$$y(t) = e^{z_0 t}, \tag{1.7}$$

where z_0 is a root of the special entire function $F(z) = 1 + \beta e^{-z\tau}$ (called below a quasi-polynomial). One sees that the system (1.6) is stable if the roots of the entire function $F(z)$ all lie in the left half plane. The above function $F(z)$ belongs to a more general class of quasi-polynomials defined next.

DEFINITION 1.2. *Let $f_0(x), \dots, f_m(x)$ be polynomials. A function of the form*

$$F(x) = f_0(x) + e^{-xT_1} f_1(x) + \dots + e^{-xT_m} f_m(x)$$

is called a quasi-polynomial*. (Pontryagyn [P42] used the term* quasi-polynomials *for the more narrow class of functions of the form* $F(x) = G(x, e^x)$*, where* $G(x, z)$ *is a polynomial in two variables.)* Let us now consider a more practical example giving rise to quasi-polynomials.

EXAMPLE 2. *Many problems in control engineering involve (multiple) time delays modelled by*

$$\frac{dy}{dt} = Ay(t) + \sum_{r=1}^{p} By(t - \tau_r). \tag{1.8}$$

This model can be construed as a representative dynamics of full state feedback systems with multiple computational and actuation delays τ_r*. The dynamics in (1.8) is also called retarded time delay system, because the highest order derivative terms are not affected by the delays. After the Laplace transformation one gets the characteristic equation*

$$F(s) = \det(sI - A - \sum_{r=1}^{p} B_r e^{-\tau_r s}) = 0$$

giving rise to the quasi-polynomial of the form

$$F(s) = f_0(s) + e^{sT_1} f_1(s) + \cdots + e^{sT_m} f_m(s) \tag{1.9}$$

where $f_k(s)$ *are polynomials. Again, the stability of the feedback system with multiple delays (1.8) is equivalent to the location of all the roots of the entire function* $F(s)$ *in the left half plane.*

The first results on stability of quasi-polynomials and entire functions were obtained in the pioneering works of Pontryagin [P42] and of Chebotarev-Meiman [CM49]. They were motivated by **(a)** controller design [specifically, of regulators and servomechanisms driving the tracking error $e(t)$ to zero]; **(b)** modelling a hit in a pipeline. The more recent relevant literature is not exhaustive, and reading [BC63], [L80], [HVL93], [BCK95], [L96], [KZ97], [DV98], [OS02] [DHB03], [KTMOM03] gives an introduction in the current state of art in this area [see also the references therein, no possible omissions are intentional].

1.3. Main results and the structure of the paper. A very close problem of studying the stability of certain "perturbed" families of entire functions was addressed, e.g., in [BCK95], [DHB03], [KTMOM03]. However, the flavor of their results seems to be different. Firstly, they considered "uncertain families" that are not *interval perturbations* as in (1.2). Secondly, their sufficient conditions[4] are are not of the form (1.3), and moreover their number can be large.

The main result of this paper is different: it is a direct generalization of the Kharitonov's theorem. In particular, we consider exactly the stability of *interval entire functions* of the type (1.2), and we show that fulfillment of the four conditions exactly of the type (1.3) suffices for stability.

The paper is structured as follows. In Sec 2 we lay the groundwork for further generalizations by recalling the standard proof of the Kharitonov theorem based on the classical Hermite-Biehler criterion [B1879]. Simple examples are included to recall that the latter criterion cannot be carried over to arbitrary entire functions. The section identifies two difficulties to overcome in deriving such a generalization.

[4]Such conditions involve certain hyperplanes [BCK95], or they are of the so-called some vertex-type [KTMOM03].

In Sec 3 we indicate how to overcome the above difficulties.

In Sec 4 we use the results of the two preceeding sections to formulate a generalization of the Kharitonov theorem for a class of entire functions.

Sec 5 contains proofs of the main results.

In Sec 6 we suggest a technique that is useful for solving some Kharitonov-like problems for matrix polynomials and matrix entire functions.

2. Two difficulties.

2.1. The Hermite-Biehler theorem and the proof of the polynomial Kharitonov theorem. First, we recall the classical Hermire-Biehler theorem, it plays an important role in establishing the Kharitonov's result for the polynomials and in extending it to entire functions.

THEOREM 2.1. *Let $F(z)$ be a polynomial in (1.1) and define two polynomials, and even $F_e(z)$, and an odd $F_o(z)$ by*

$$F(z) = F_e(z) + F_o(z), \tag{2.1}$$

and denote

$$P(z) = F_e(jz), \qquad Q(z) = \frac{F_o(jz)}{j}.$$

Then the polynomial $F(z)$ is stable if and only if the following two conditions hold true.

1. *The roots of the polynomials $P(z)$ and $Q(z)$ are all real and they interlace.*

2. *There is at least one point $z_0 \in \mathbb{R}$ such that*

$$P(z_0)Q'(z_0) - P'(z_0)Q(z_0) > 0. \tag{2.2}$$

REMARK 2.

- *The condition 1 is equivalent to the fact that that the roots of the polynomials $F_e(z)$ and $F_o(z)$ are all purely imaginary and they interlace.*
- $$\Phi(z) = F(jz) = P(z) + jQ(z). \tag{2.3}$$
- *If the (2.2) is fulfilled just for one point $z_0 \in \mathbb{R}$ then it is valid for all $z \in \mathbb{R}$.*
- *For the real $F(z)$ the condition (2.2) simply means that $f_n \cdot f_{n-1} > 0$, where f_n, f_{n-1} are the coefficients of $F(z)$ in (1.1).*

The proof of the Kharitonov theorem is now immediate. Figure 1 (that illustrates the remark 1 and theorem 2.1) shows that any interval perturbation of the odd terms gives us a stable polynomial. Fixing the perturbed odd part and applying a similar argument to the even part one finally obtains the theorem 1.1.

In the next two subsections we show that there are at least two problems in carrying over the above arguments to arbitrary entire functions.

2.2. The first difficulty. The fixed degree property and the number of roots. Figure 1 indicates that interval perturbations do not destroy interlacing of the roots of $F_e(x)$ and $F_o(x)$. However, it is implicitly assumed that the all polynomials $F(z)$ in (1.1) and (1.2) have the same degree. Hence the number of the roots of each of the polynomials $F_e(x)$ and $F_o(x)$ in (2.1) stays the same.

However, if $F(x)$ were an entire function then its degree is "infinite," and hence one has to take care of preventing new roots from occurring. Graphically, one has to prevent the phenomenon shown in Figre 2 by means of *imposing certain additional constrains* analogous to the fixed-degree property.

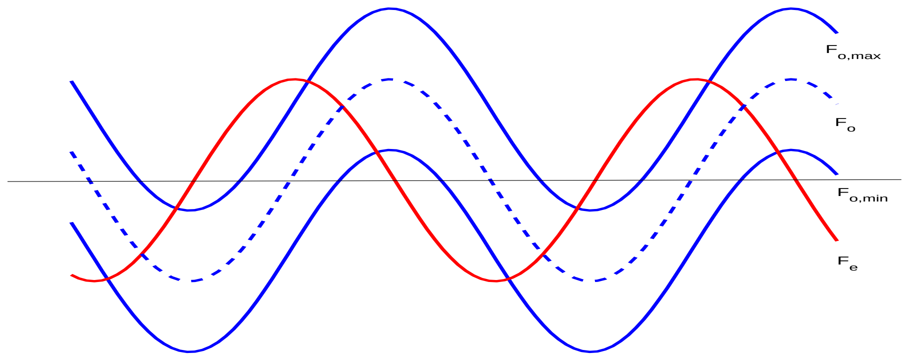

FIG. 2.1. Illustration for the Proof of the classical Kharitonov theorem for polynomials via the Hermite-Biehler.

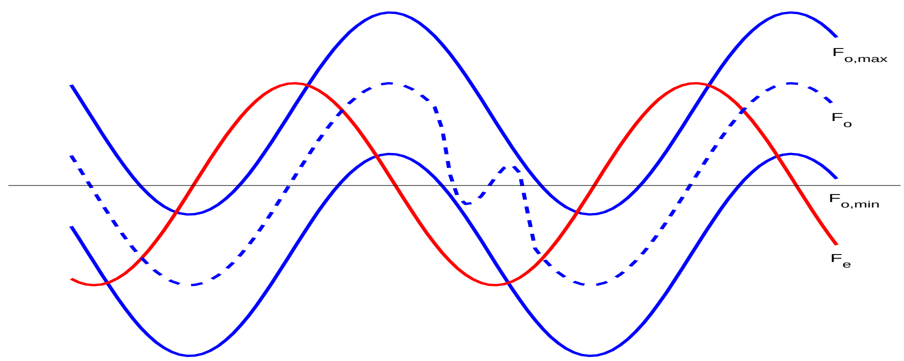

FIG. 2.2. An illustration for the first difficulty. One has to impose additional conditions preventing arising of the new roots of F_o.

2.3. The second difficulty. The Hermite-Biehler theorem does not carry over to entire functions. EXAMPLE 3. **Interlacing of the roots is not necessary.** *The function*

$$\Phi(z) = e^{jz^2},$$

does not have roots at all and hence stable. However, using the definition (2.3) we see that

$$P(z) = \cos z^2, \qquad Q(z) = \sin z^2,$$

have non-real roots, e.g., $Q(j\sqrt{\pi}) = 0$.

EXAMPLE 4. **Interlacing of the roots is not sufficient.** *Consider*

$$\Phi(z) = e^{jz} + je^{-jz}.$$

Again, using the definition (2.3) we see that

$$P(z) = e^z, \qquad Q(z) = e^{-z}$$

do not have roots at all, and hence the interlacing property is fulfilled. Moreover,

$$P(z)Q'(z) - P'(z)Q(z) = 2$$

for any w. Nevertheless $\Phi(\frac{3}{4}\pi) = 0$.

Again, the second difficulty also indicates that it is hopeless to extend the Kharitonov's theorem to the class of all entire functions. Similarly to the remark made at the end of the subsection 2.2, the challenge here is to identify a class of entire functions for which the two difficulties can be removed, and then to try to prove the Kharitonov's theorem for that class. This is precisely what is done in the rest of the paper.

3. Removing the two difficulties.

3.1. Removing the First Difficulty. An analogue of the fixed degree property. Recall that even for polynomials the conditions

$$\deg F_o(x) = \deg F_{o,max}, \quad \deg F_e(x) = \deg F_{e,max}$$

are implicitly included in the classical Kharitonov theorem and they are crucial for its validity. (Here we used the definition in (2.3).) In our generalized Kharitonov theorem these conditions will be included in the following form (that applies not only to polynomials but to entire functions as well):

$$\frac{F_o(z)}{F_{o,max}(z)} = O(1), \qquad \frac{F_e(z)}{F_{e,max}(z)} = O(1), \qquad (z \in \mathbb{R})$$

and

$$0 < m_o \leq \left| \frac{F_{o,min}(z)}{F_{o,max}(z)} \right| \leq M_o < \infty, \qquad 0 < m_e \leq \left| \frac{F_{e,min}(z)}{F_{e,max}(z)} \right| \leq M_e < \infty, \qquad (z \in \mathbb{R}).$$

All latter expressions make sense for any $z \in \mathbb{R}$ since all the roots of $F_{o,max}(z)$, $F_{e,max}(z)$ are purely imaginary, cf. with the remark 2.

3.2. Removing the Second Difficulty. The class HP .

3.2.1. Classical results. The class P. In this subsection we recall the basic definitions and results that can be found in [CM49] and [L80], [L96]. They will be used in what follows. We discuss here the situation in which $\Phi(z)$ does not have roots in the lower half plane. In the next sections these settings will be adjusted for the stability of $F(z) = \Phi(jz)$.

DEFINITION 3.1. *Let $\Phi(z)$ be an entire function of finite exponential type.*

- *To characterize the growth of $\Phi(z)$ Phragmén and Lindelöf introduced the function*

$$h_\Phi(\theta) = \overline{\lim_{r\to\infty}} \frac{\log|\Phi(re^{i\theta})|}{r}, \qquad \theta \in \mathbb{R}. \tag{3.1}$$

  which is called an **indicator function** *of $\Phi(z)$.*
- *The quantity*

$$d_\Phi = h_\Phi(-\frac{\pi}{2}) - h_\Phi(\frac{\pi}{2})$$

  is called the **defect** *of the function $\Phi(z)$.*

- *An entire function of finite exponential type is said to be in the* **class P** *if it has no zeros in the open lower half-plane and*

$$d_\Phi \geq 0. \tag{3.2}$$

The indicator function h_Φ plays a crucial role on the rest of the paper. One reason is that its behavior is connected with interlacing of the roots, as described next.

THEOREM 3.2. *([CM49], [L96], [K]) Let us partition the entire function of finite exponential type*

$$\Phi(z) = P(z) + jQ(z) \tag{3.3}$$

so that $P(z), Q(z)$ are real entire functions. $\Phi(z)$ belongs to the class P if and only if

1. *the roots of $P(z)$ and $Q(z)$ are all real and interlacing;*
2. *the indicator functions of $P(z)$ and $Q(z)$ coincide:*

$$h_P(\theta) = h_Q(\theta); \tag{3.4}$$

3. *at some real point z_0 we have*

$$Q'(z_0)P(z_0) - Q(z_0)P'(z_0) > 0. \tag{3.5}$$

(If the latter condition is fulfilled just for one point $z_0 \in \mathbb{R}$ then it is valid for all $x \in \mathbb{R}$.)

3.2.2. A slight modification. The class HP. Application to stability. It is convenient to explicitly translate the results of Sec 3.2.1 to the left half plane case (stability) and to use them thereafter.

DEFINITION 3.3. *Let $F(z)$ be an entire function of finite exponential type.*

- *We suggest to refer to the quantity*

$$d_F^{(HP)} = h_F(0) - h_F(\pi)$$

as the **HP-defect** *of the function $F(z)$.*

- *We shall reserve the name the* **HP class** *(i.e., Hurwitz P) for entire functions of finite exponential type with no zeros in the open left half-plane and satisfying*

$$d_F^{(HP)} \geq 0. \tag{3.6}$$

Formula (2.3) yields the following corollary.

THEOREM 3.4. *(cf. with theorem 3.2) Let us partition the real entire function of finite exponential type*

$$F(z) = \underbrace{F_e(z)}_{\text{even degree terms}} + \underbrace{F_o(z)}_{\text{odd degree terms}}. \tag{3.7}$$

$F(z)$ belongs to the class HP if and only if

1. *the roots of $F_e(z)$ and $F_o(z)$ are all* purely imaginary *and interlacing;*
2. *the indicator functions of $F_o(z)$ and $F_e(z)$ coincide:*

$$h_{F_e}(\theta) = h_{F_o}(\theta); \tag{3.8}$$

3. We have

$$F_o'(0)F_e(0) > 0. \tag{3.9}$$

Theorem 3.4 yields the following result.

THEOREM 3.5. *(cf. with [L80], Ch. VI, sec. 4. thm 8.) Let $F(z)$ be an entire function of finite exponential type with no roots on the imaginary axis satisfying*

$$d_F^{(HP)} > 0. \tag{3.10}$$

$F(z)$ is stable if and only if the roots of $F_o(z)$ and $F_e(z)$ in (3.7) all lie on the imaginary axis and interlace.

EXAMPLE 5. **Stability of quasi-polynomials.** *Consider*

$$F(z) = \sum_1^m e^{\lambda_k z} f_k(z), \tag{3.11}$$

where $f_k(z)$ are real polynomials, and $\lambda_1 < \lambda_2 < \ldots < \lambda_n$. Then

$$h_F(\theta) = \begin{cases} \lambda_n \cos\theta & -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\ -\lambda_1 \cos\theta & \frac{\pi}{2} \le \theta \le \frac{3\pi}{2} \end{cases}$$

If we assume $|\lambda_1| < \lambda_n$ then $h_F(0) = \lambda_n, \qquad h_F(\pi) = \lambda_1$, i.e.,

$$d_F^{(HP)} = \lambda_n - \lambda_1 > 0. \tag{3.12}$$

In accordance with theorem 3.5, in this example interlacing of the roots of $F_o(x)$ and $F_e(x)$ (that are all purely imaginary) is the necessary and sufficient condition[5] *for the stability of $F(z)$.*

EXAMPLE 6. *Let*

$$F(z) = \sinh(z)f_1(z) + \cosh(z)f_2(z),$$

where $f_1(z), f_2(z)$ are polynomials. In this case $d_F^{(HP)} = 0$.

In the latter example the condition (3.10) is not fulfilled which shows that the theorem 3.5 is not universal. Hence to derive a generalization of the Kharitonov theorem we may need to use theorem 3.4, e.g., its condition (3.8). This is a subject of the next subsection.

3.2.3. Removing the Second Difficulty. The class HP.. The class of functions satisfying (3.6) is a natural generalization of polynomials. Indeed, the definition (3.10) immediately yields that if $F(z)$ is a polynomial then $h_F(\theta) \equiv 0$ and hence $d_F^{(HP)} \equiv 0$.

Hence, the class of *stable polynomials* coincides with polynomials belonging to *class HP*. Therefore, in the context of the generalization of the Kharitonov theorem to entire functions it is natural to replace stability by belonging to the class HP.

4. The generalized Kharitonov theorem.

[5] Alternatively, all the roots of $P(z) = F_e(jz)$ and $Q(z) = \frac{F_o(jz)}{j}$ are real and interlace.

4.1. Main results. Throughout this section we use a decomposition

$$F(z) = \underbrace{F_e(z)}_{\text{even degree terms}} + \underbrace{F_o(z)}_{\text{odd degree terms}} . \tag{4.1}$$

of an arbitrary entire function $F(z)$ satisfying

$$\overline{F(\overline{z})} = F(z).$$

LEMMA 4.1. *Let*

$$F_{min}(z) = F_e(z) + F_{o,min}(z), \tag{4.2}$$

$$F_{max}(z) = F_e(z) + F_{o,max}(z) \tag{4.3}$$

be two entire functions of finite exponential type that belong to the class HP and satisfy

$$0 < m_o \leq \left| \frac{F_{o,min}(z)}{F_{o,max}(z)} \right| \leq M_o < \infty \qquad (z \in \mathbb{R}), \tag{4.4}$$

Then all the functions of the form

$$F(z) = F_e(z) + F_o(z)$$

satisfying[6]

$$\frac{F_{o,min}(jz)}{jz} \leq \frac{F_o(jz)}{jz} \leq \frac{F_{o,max}(jz)}{jz} \qquad (z \in \mathbb{R}) \tag{4.5}$$

and

$$\frac{F_o(z)}{F_{o,max}(z)} = O(1) \qquad (z \in \mathbb{R}) \tag{4.6}$$

simultaneously belong to the class HP.

We shall prove the above lemma at the end of this section.The result dual to the lemma 4.1 is stated next.

LEMMA 4.2. *Let*

$$F_{min}(z) = F_{e,min}(z) + F_o(z), \tag{4.7}$$

$$F_{max}(z) = F_{e,max}(z) + F_o(z) \tag{4.8}$$

be two entire functions of finite exponential type that belong to the class HP and satisfy

$$0 < m_e \leq \left| \frac{F_{e,min}(z)}{F_{e,max}(z)} \right| \leq M_e < \infty \qquad (z \in \mathbb{R}), \tag{4.9}$$

Then all the functions of the form

$$F(z) = F_e(z) + F_o(z)$$

[6]This is an analog of the interval uncertainty (1.4).

with

$$F_{e,min}(jz) \le F_e(jz) \le F_{e,max}(jz) \qquad (z \in \mathbb{R}), \tag{4.10}$$

and

$$\frac{F_e(z)}{F_{e,max}(z)} = O(1) \qquad (z \in \mathbb{R}) \tag{4.11}$$

simultaneously belong to the class HP.

The lemmas 4.1 and 4.2 imply the following result.

THEOREM 4.3. **Generalized Kharitonov theorem.** *Let the conditions (4.4), (4.5), (4.6), and (4.9), (4.10), (4.11) be fulfilled.*

If only four functions

$$F_{min,min}(z) = F_{e,min}(z) + F_{o,min}(z), \qquad F_{min,max}(z) = F_{e,min}(z) + F_{o,max}(z)$$

$$F_{max,min}(z) = F_{e,max}(z) + F_{o,min}(z), \qquad F_{max,max}(z) = F_{e,max}(z) + F_{o,max}(z)$$

belong to the class HP then all the functions

$$F(z) = F_e(z) + F_o(z)$$

belong to the class HP as well.

REMARK 3. *Recall that for polynomials $F(z)$ we have $d_F^{(HP)} \equiv 0$ so that the class HP is simply the class of stable polynomials. Hence theorem 4.3 is a direct generalization of the Kharitonov theorem 1.1.*

REMARK 4. *Recall that for the quasi-polynomials $F(z)$ the value $d_F^{(HP)}$ is given by the closed form expression (3.12). Hence it is often possible to see that the perturbations (4.5) and (4.10) yield a family satisfying the condition (3.10). If it is the case then all the results of this subsection are valid in a stonger formulation, i.e., in which one replacea the class HP by the class of stable quasi-polynomials.*

4.2. Some examples. EXAMPLE 7. **Chebotarev and Meiman.** *Addresing a mechanical problem suggested by Voznesensky Chebotarev and Meiman [CM49] considered the function*

$$F(z) = e^z q(z) + p(z),$$

where $q(z)$ and $p(z)$ were polynomials of degree 5. Let us consider here the more general case with

$$\deg q(z) \ge \deg p(z).$$

Thinking of $q(z)$ as of a fixed polynomial, and of $p(z)$ as of an interval polynomial we construct for $p(x)$ the four corresponding polynomials $p_1(z)$, $p_2(z)$, $p_3(z)$, $p_4(z)$ by the Kharitonov rule.

Then the following assertion is valid. If the four functions

$$F_k(z) = e^z q(z) + p_k(z),$$

are stable then all $F(z)$ are stable, too.

EXAMPLE 8. **Quasi-polynomials.** *Let*

$$F(z) = e^{\lambda_1 z} q(z) + e^{\lambda_2 z} p(z) + e^{\lambda_3 z} r(z), \tag{4.12}$$

where $p(z), q(z), r(z)$ *are polynomials satisfying*

$$\deg r(z) \geq \max\{\deg p(z), \deg q(z)\},$$

and

$$\lambda_3 > \lambda_2 > \lambda_1 \geq 0, \qquad \lambda_3 + \lambda_1 \geq 2\lambda_2.$$

Again, let us thinking of $q(z)$ *az of a fixed polynomial, and of* $p(z)$ *and* $r(z)$ *as of an interval polynomials, and let us construct the four polynomials* $p_k(z)$ *and the four polynomials* $r_k(z)$ *using the Kharitonov recipe.*

PROPOSITION 4.4. *If the sixteen functions*

$$F_{k,l}(z) = e^{\lambda_1 z} q(z) + e^{\lambda_2 z} p_k(z) + e^{\lambda_3 z} r_l(z)$$

are stable than all of the quasi-polynomial $F(z)$ *in (4.12) are stable.*

5. Proof of the lemma 4.1.. Since both $F_{min}(z)$ and $F_{max}(z)$ belong to the class HP, the roots of each of the $F_{o,min}(z)$ and $F_{o,max}(z)$ interlace with the ones of $F_e(z)$ by theorem 3.4. Now, (4.5) implies the following statement. Between any two successive zeros of $F_e(z)$ there could be either one (cf. with figure 2.1) or more (cf. with figure 2.2) zeros of $\frac{F_o(z)}{z}$. We prove that there is always only one zero. Denote the positive roots of the following functions as follows:

functions after $z \leftarrow jz$	their (real) roots
$F_e(jz)$	$\{a_k\}$
$\frac{F_{o,min}(jz)}{jz}$	$\{b_{1k}\}$
$\frac{F_o(jz)}{jz}$	$\{b_k\}$
$\frac{F_{o,max}(jz)}{jz}$	$\{b_{2k}\}$

so that we have

$$0 < a_k < b_{1k} \leq b_k \leq b_{2k} < a_{k+1}. \tag{5.1}$$

We do not know yet that there is only one b_k between a_k and a_{k-1}. So let us think for a moment that there are more, but we work with only a subset $\{b_k\}$ of the roots of $F_o(jz)/jz$ choosing only one such b_k between a_k and a_{k-1}. Since the roots of the functions $F_{o,min}(z), F_{o,max}(z)$ are imaginary and symmetric with respect to the origin, their Hadamard decompositions [L96] have the form

$$F_{o,min}(z) = c_1 z \prod_{k=1}^{\infty} (1 + \frac{z^2}{b_{1k}^2}), \tag{5.2}$$

$$F_{o,max}(z) = c_2 z \prod_{k=1}^{\infty} (1 + \frac{z^2}{b_{2k}^2}). \tag{5.3}$$

Let us construct the function

$$\tilde{F}_o(z) = cz \prod_{k=1}^{\infty}(1 + \frac{z^2}{b_k^2}), \tag{5.4}$$

where

$$c = \tilde{F}_o{}'(0). \tag{5.5}$$

It follows from (4.5) that

$$c_1 \le c \le c_2. \tag{5.6}$$

Since the $\{b_k\}$ are the subset of all the roots of $F_o(z)$ hence

$$F_o(z) = \tilde{F}_o(z) \cdot R(z), \tag{5.7}$$

where $R(z)$ is an entire function. Now, the relations (4.4), (4.6), and (5.2) - (5.4) imply

$$0 < m_o \le \mid \frac{F_o(z)}{\tilde{F}_o(z)} \mid \le M_o < \infty \quad \text{for} \quad z = \overline{z}. \tag{5.8}$$

Clearly, (5.7) and (5.8) mean that $R(z) = \frac{F_o(z)}{\tilde{F}_o(z)}$ is bounded on $z = \overline{z}$. Interlacing of the roots of $\tilde{F}_o(z)$ and $F_e(z)$ and lemma 2 of sec 27.3 in [L96] imply

$$h_{\tilde{F}_o} = h_{F_e}. \tag{5.9}$$

Similarly, interlacing of the roots of $F_o(z)$ and $F_e(z)$ [L96] implies

$$h_{F_o} = h_{F_e}. \tag{5.10}$$

The two latter equations (5.9), (5.10) yield

$$h_{\tilde{F}_o} = h_{F_o}. \tag{5.11}$$

Now, all the roots $\{z_k\}$ of $\tilde{F}_o(z)$ are purely imaginary and

$$\lim_{r\to\infty} \sum_{|z_k|<r} \frac{1}{z_k} = 0.$$

Hence the function $\tilde{F}_o(z)$ by the result of Lecture 5 in [L96] must have the completely regular growth[7] Since one of the factors in (5.7) has completely regular growth, by theorem 5 of Chapter III in [L80] we have

$$h_{F_o}(\theta) = h_{\tilde{F}_o}(\theta) + h_R(\theta), \qquad \text{i.e.,} \qquad h_R(\theta) = 0.$$

[7] An entire function is referred to as a function of *completely regular growth* if the function

$$h_F(r,\theta) = \frac{ln|F(re^{j\theta}|}{r}$$

uniformly converges to $h_F(\theta)$ for $r \to \infty$ almost everywhere.

The latter equation mean that the function $R(z)$ is of the minimal type. By the Phragmén-Lindelöf principle [L96] $R(z)$ is bounded everywhere entire function and hence a constant. In view of (5.4) and

$$\lim_{z\to 0}\frac{F_o(z)}{z}=\lim_{z\to 0}\frac{\tilde{F}_o(z)}{z}$$

we have $R(0)=1$. Hence the roots of $F_o(z)=\tilde{F}_o(z)$ do interlace with the ones of $F_e(z)$ yielding the condition 1) of theorem 3.4.

Secondly, the condition (3.8) was established in follows from (5.10).

Finally, (3.9) is fulfilled thanks to (5.5) and expressions (5.2), (5.3), (5.4).

Hence $F(z)$ belong to the class HP by theorem 3.4, and the lemma is proven.

6. The matrix case. In many applied problems quasi-polynomials occur as characteristic polynomials of a certain system. However, the interval family the original systems is not translated immediately into the interval family of the corresponding quasi-polynomials.

Here we suggest a certain alternative, namely to try to construct the "edge" polynomials for the interval family of quasi-polynomials directly in terms of the original system. We first formulate the general suggestion, and then consider certain examples.

6.1. A general suggestion. Let $\mathcal{P}(z)$ is an arbitrary $m\times m$ entire function such that

$$\mathcal{P}^*(\overline{z})=\mathcal{P}(z).$$

Again, we represent

$$\mathcal{P}(z)=\underbrace{\mathcal{P}_e(z)}_{\text{even degree terms}}+\underbrace{\mathcal{P}_o(z)}_{\text{odd degree terms}}. \tag{6.1}$$

Now, suppose that one succeeded to construct four matrices

$$\mathcal{P}_{e1}(z),\qquad \mathcal{P}_{e2}(z),\qquad \mathcal{P}_{o1}(z),\qquad \mathcal{P}_{o2}(z),$$

such that for all $m\times 1$ vectors h the *scalar* quasi-polynomials

$$F_{ok}(z,h)=h^*\mathcal{P}_{ok}(z)h,\qquad k=1,2$$

$$F_{ek}(z,h)=h^*\mathcal{P}_{ek}(z)h,\qquad k=1,2$$

$$P(z,h)=h^*\mathcal{P}(z)h$$

satisfy the conditions of the generalized Kharitonov theorem. Then $\mathcal{P}(z)$ is a stable matrix function.

6.2. Some examples. EXAMPLE 9. **Retarded differential equation.** *Consider*

$$\frac{d}{dt}x(t)+ax(t)+Bx(t-\tau)=0,$$

where $x(t)$ is an $m\times 1$ vector function and B is an $m\times m$ matrix, and a is a number. The characteristic matrix of this equation has the form

$$\Delta(z)=(z+a)I_m+Be^{-z}=e^{-z}\mathcal{P}(z),$$

where

$$\mathcal{P}(z) = (z+a)e^z I_m + B, \tag{6.2}$$

We suppose that there exist two numbers b_-, b_+ such that

$$b_- I_m \le B \le b_+ I_m.$$

PROPOSITION 6.1. *The matrix function $\mathcal{P}(x)$ is stable if*

$$a > -1, \qquad a + b_\pm > 0, \qquad b_\pm < \xi \sin\xi - a\cos\xi,$$

where $\xi = \frac{\pi}{2}$ if $a = 0$ and if $a \neq 0$ then ξ is the root of the equation

$$\xi = -a\tan\xi, \qquad 0 < \xi < \pi$$

if $a \neq 0$.

Let us prove the latter proposition. Indeed, using a result of Hyes [H50] (see also [HVL93]) we deduce that the functions

$$p_\pm = (a+z)e^z + b_\pm$$

are all stable. The proof follows immediately from the matrix version of the generalized Kharitonov theorem.

EXAMPLE 10. *Let us consider the $m \times m$ matrix function*

$$\mathcal{P}(z) = z^2 e^z I_m + Az + B$$

where A and B are $m \times m$ matrices satisfying

$$a_- I_m \le A \le a_+ I_m, \qquad b_- I_m \le B \le b_+ I_m$$

with certain numbers a_-, a_+, b_-, b_+.

PROPOSITION 6.2. *The matrix function $\mathcal{P}(z)$ is stable if*

$$0 < a_- < a_+ < \frac{\pi}{2},$$

$$0 < b_- < b_+ < \min\{\alpha_+^2 \cos\alpha_+, \ \ \alpha_-^2 \cos\alpha_-\}.$$

where $\alpha_\pm$ are the roots of

$$\sin\alpha_\pm = \frac{a_\pm}{\alpha_\pm}, \qquad 0 < \alpha_\pm < \frac{\pi}{2}.$$

The proof follows immediately from the matrix version of the generalized Kharitonov theorem and the fact that all the functions

$$\begin{aligned} p_1(z) &= z^2 e^z + a_- z + b_-, \\ p_2(z) &= z^2 e^z + a_+ z + b_+, \\ p_3(z) &= z^2 e^z + a_- z + b_+, \\ p_4(z) &= z^2 e^z + z_+ z + b_- \end{aligned} \tag{6.3}$$

are stable. The stability of the four scalar quasi-polynomials in (6.3) follows from 13.9 of [BC63], and moreove, the result of the above proposition makes sense even in the simplest scalar case.

[B1879] M. Biehler, *Sur une classe déquations algébriques dont toutes les racines sont réelles*, J. Reine Angew. Math. 87 (1879) 350352.

[BC63] R.Bellman and K.Cooke, *Differential-Difference Equations*, Academic Press, 1963.

[BCK95] S.P. Bhattacharyya, H. Chapellat, L.H. Keel, *Robust Control - The Parametric Approach*, Prentice Hall, 1995.

[CM49] N.Chebotarev, N.Meiman, *The Routh-Hurwitz probelm for polynomials and entire functions,* Trudy MIAN, 1949, vol. 26.

[DHB03] A.Datta, M.-T. Ho and S.P. Bhattacharyya, *Structure and Synthesis of PID Controllers*, Springer Verlag, 2003.

[DV98] L.Dugard and E.Verriest (eds), *Stability and control of time-delay systems*, Springert Verlag 1998,

[F53] S. Faedo, *Un nuovo problema di stabilita per le equazioni algebriche e coefficient reali,* Ann. Scuola normale superiore, Pisa, Sci. fos. mat., 1953, vol, 7, no 1-2. p. 53-63

[H1856] Hermite, C., *Extrait d'une lettre de Mr. Ch. Hermite de Paris à Mr. Borchardt de Berlin, sur le nombre des racines d'une èquation algèbrique comprises entre des limits donèes,* J. Reine Angew. Math., **52** (1856), 39-51.

[H1895] A. Hurwitz, *Über die Bedingungen, unter welchen eine Gleichung nurWurzeln mit negativen reellen Teilen besitzt*, Math. Ann. 46 (1895) 273284.

[H50] N.D.Hayes, *Roots of the transcedental equation associated with a certain differential difference equations*, J. London Math. Society, 25 (1950), 226-232.

[HVL93] J.K. Hale and S.Verdun Lunel, *Introduction to Functional Differential Equations*, Springer-Verlag, New York, Applied Mathematical Sciences Vol. 99, 1993.

[K] M.G.Krein. As remarked in [L96] the theorem 3.2 was also obtain by M.G.Krein, but it was never published by him.

[K78] Kharitonov V.L., *Asymptotic stability of an equilibrium position of a family of systems of differential equations,* Differenzyal'nye uravneniya, **14** (1978), 2086-2088.

[KTMOM03] Kharitonov, V.L.; Torres-Munoz, J.A.; Ortiz-Moctezuma, M.B.; *Polytopic families of quasi-polynomials: vertex-type stability conditions*, Circuits and Systems I: Fundamental Theory and Applications, IEEE Transactions on [see also Circuits and Systems I: Regular Papers, IEEE Transactions on] , Volume: 50 , Issue: 11 , Nov. 2003 Pages:1413 - 1420

[KZ97] Kharitonov, V.L. and Zyabko A., *Robust Stability of Time Delay Systems*, IEEE Trans. on Automatic Control, 39 (1994), 2388-2397.

[L80] B.Ya.Levin. Distribution of zeros of entire functions. Revised edition. Amer. Math. Soc. Providence, RI, 1980.

[L96] B.Ya. Levin , *Lectures on Entire Functions ,* AMS publications, 1996.
online version: http://www.ams.org/online_bks/mmono150/

[M1868] J. Maxwell. *On Governors,* Proceedings of the Royal Society, No. 100. 1868

[OS02] Nejat Olgac and Rifat Sipahi, *An Exact Method for the Stability Analysis of Time-Delayed Linear Time-Invariant (LTI) Systems*, IEEE Trans. on Automatic Control, vol. 47, no. 5, May 2002, 793-797.

[OS03] V.Olshevsky and L.Sakhnovich, *An Operator Identities Approach to Bezoutians. A General Scheme and Examples*, 2003, preprint.

[P42] L.Pontryagin, *On the zeros of some transcedental functions,* IAN USSR, Math. series, vol. 6, 115-134, 1942.

[R1977] E.J. Routh, *Stability of a given state of motion*, London, 1877.

[VL02] S. Verdun Lunel, *Spectral Thgeory for neutral delay equations with applications to control and stabilization*, in "Math. Systems Theory in Biology, Communications and Finance," (J.Rosenthal, D.Gillivan, eds), p. 415-469 Springer, 2002.